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On a Class of Equations of Transformation.

BY JACOB WESTLUND.

The object of the following paper is to discuss those equations of transformation whose roots are the $n+1$ values of

$$y_{\frac{\mu}{\nu}} = \prod_{1,\,m}^{p} \mathrm{sn}^{2\alpha}.\,\mathrm{cn}^{\beta}.\,\mathrm{dn}^{\gamma}\,(4p\varpi/k),$$

where α, β, γ are any positive or negative integers and $\varpi = \dfrac{4\mu\mathrm{K}+4\nu i\mathrm{K}'}{n}$, μ and ν being integers.

We use the notation $y_{\frac{\mu}{\nu}}$, since the value of the roots depends on the *ratio* of the integers μ and ν.* We also suppose, for simplicity, n to be an odd prime and set $n = 2m+1$.

The equation in question being rational for the domain of rationality, $R(\varkappa)$, where $\varkappa = k^2$, may be written

$$f_0(\varkappa)\,y^{n+1} + f_1(\varkappa)\,y^n + \ldots\ldots + f_{n+1}(\varkappa) = 0\,,$$

where
$$f_a(\varkappa) = a_0 + a_1\varkappa + \ldots\ldots + a_{m_a}\varkappa^{m_a} \tag{1}$$

The method for computing the coefficients $f_0 \ldots\ldots f_{n+1}$ is similar to that applied by Pierpont† to the modular equation, and consists of the following three steps:

1. To express the roots as q-series.
2. To find a superior limit of the degree in $\varkappa$ of the equation.

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*Weber, "Elliptische Functionen," §67.

† "On Modular Equations" (Bulletin Am. Math. Soc., vol. III, No. 8, 1897).

3. To express $\varkappa$ as a q-series and substitute for y one of the roots expressed as a q-series in the equation and equate the coefficients of the different powers of q to zero, by which we obtain a system of linear equations for determining the constants that enter into the coefficients $f_0 \ldots . f_{n+1}$.

1.—*To Develop the Roots into q-series.*

If we set

$$u_{\frac{\mu}{\nu}} = \prod_{1,\,m}^{p} \operatorname{sn}^2(4p\varpi),$$

$$v_{\frac{\mu}{\nu}} = \prod_{1,\,m}^{p} \operatorname{cn}(4p\varpi),$$

$$w_{\frac{\mu}{\nu}} = \prod_{1,\,m}^{p} \operatorname{dn}(4p\varpi),$$

we have

$$y_{\frac{\mu}{\nu}} = u^{\alpha}_{\frac{\mu}{\nu}} \,.\, v^{\beta}_{\frac{\mu}{\nu}} \,.\, w^{\gamma}_{\frac{\mu}{\nu}} \,.$$

I.—$u_{\frac{\mu}{\nu}}$.

Using the formula*

$$\prod_{1,\,m}^{p} \frac{\vartheta_1(v-p\omega)\,\vartheta_1(v+p\omega)}{\vartheta_0(v-p\omega)\,\vartheta_0(v+p\omega)} = (-1)^{\frac{n_1-1}{2}} \,.\, \frac{\vartheta_1}{\vartheta_0}\left(v_{\frac{\mu}{\nu}}/\tau_{\frac{\mu}{\nu}}\right) \frac{\vartheta_0}{\vartheta_1}(v/\tau), \tag{2}$$

where

$$\tau_{\frac{\mu}{\nu}} = \frac{\tau + 16a}{n}, \quad v_{\frac{\mu}{\nu}} = v \;, \quad n_1 = 1, \quad \text{if } \nu \neq 0,$$

$$\tau_{\frac{\mu}{\nu}} = n\tau \qquad , \quad v_{\frac{\mu}{\nu}} = nv, \quad n_1 = n, \quad \text{if } \nu = 0,$$

and

$$\omega = \frac{\mu + \nu\tau}{n},$$

and observing that

$$\lim_{v=0} \frac{\vartheta_1\left(v_{\frac{\mu}{\nu}}/\tau_{\frac{\mu}{\nu}}\right)}{\vartheta_1(v/\tau)} = n_1 \frac{\vartheta_0\left(0/\tau_{\frac{\mu}{\nu}}\right)\vartheta_2\left(0/\tau_{\frac{\mu}{\nu}}\right)\vartheta_3\left(0/\tau_{\frac{\mu}{\nu}}\right)}{\vartheta_0(0/\tau)\,\vartheta_2(0/\tau)\,\vartheta_3(0/\tau)}$$

we get

$$(-1)^m \prod_{1,\,m}^{p} \frac{\vartheta_1^2}{\vartheta_0^2}(p\omega) = n_1(-1)^{\frac{n_1-1}{2}} \,.\, \frac{\vartheta_2\left(0/\tau_{\frac{\mu}{\nu}}\right) . \,\vartheta_3\left(0/\tau_{\frac{\mu}{\nu}}\right)}{\vartheta_2(0/\tau) \,.\, \vartheta_3(0/\tau)} \tag{3}$$

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* Königsberger, "Vorlesungen über Ellipt. Funct.," II, pp. 96–97.

and hence

$$\left.\begin{aligned} u_\infty &= \frac{n}{k^m}\cdot\frac{\vartheta_2(0/n\tau)\,.\,\vartheta_3(0/n\tau)}{\vartheta_2(0/\tau)\,.\,\vartheta_3(0/\tau)}\,, \\ \underset{a=0,\,1,\,\dots,\,n-1}{u_a} &= \frac{(-1)^m}{k^m}\cdot\frac{\vartheta_2\left(0\Big/\dfrac{\tau+16a}{n}\right)\vartheta_3\left(0\Big/\dfrac{\tau+16a}{n}\right)}{\vartheta_2(0/\tau)\,.\,\vartheta_3(0/\tau)}\,, \end{aligned}\right\} \tag{4}$$

$$\left.\begin{aligned} u_\infty &= n\,.\,2^{-2m}(1+a_1q+\dots\dots)\,, \\ \underset{a=0,\,1,\,\dots,\,n-1}{u_a} &= (-1)^m.\,2^{-2m}.\,q^{-\frac{2r}{n}}\,.\,\varepsilon^{2a}(1+b_1\,\varepsilon^{8a}q^{\frac{1}{n}}+\dots\dots)\,, \end{aligned}\right\} \tag{5}$$

where $r=\dfrac{n^2-1}{8}$ and $\varepsilon=e^{\frac{2\pi i}{n}}$.

II.—$v_{\frac{\mu}{\nu}}$.

We have*

$$\prod_{1,\,m}^{p}\frac{\vartheta_2(v-p\omega)\,\vartheta_2(v+p\omega)}{\vartheta_0(v-p\omega)\,\vartheta_0(v+p\omega)}=\frac{\vartheta_2}{\vartheta_0}(v_{\frac{\mu}{\nu}}/\tau_{\frac{\mu}{\nu}})\cdot\frac{\vartheta_0}{\vartheta_2}(v/\tau). \tag{6}$$

Hence $$\prod_{1,\,m}^{p}\frac{\vartheta_2^2}{\vartheta_0^2}(p\omega)=\frac{\vartheta_2}{\vartheta_0}(0/\tau_{\frac{\mu}{\nu}})\cdot\frac{\vartheta_0}{\vartheta_2}(0/\tau)$$

and
$$\begin{aligned} v^2_{\frac{\mu}{\nu}} &= \left(\sqrt{\frac{k'}{k}}\right)^n\cdot\frac{\vartheta_2}{\vartheta_0}(0/\tau_{\frac{\mu}{\nu}}) \\ &= \frac{\psi^{2n}(\tau)}{\phi^{2n}(\tau)}\cdot\frac{\phi^2(\tau_{\frac{\mu}{\nu}})}{\psi^2(\tau_{\frac{\mu}{\nu}})}\,, \end{aligned}$$

where $\phi(\tau)=\sqrt[4]{k}$, $\psi(\tau)=\sqrt[4]{k'}$.

Hence
$$v_{\frac{\mu}{\nu}}=\rho_{\frac{\mu}{\nu}}\cdot\frac{\psi^n(\tau)}{\phi^n(\tau)}\cdot\frac{\phi(\tau_{\frac{\mu}{\nu}})}{\psi(\tau_{\frac{\mu}{\nu}})}\,, \tag{7}$$

where $\rho_{\frac{\mu}{\nu}}=\pm 1$.

In order to determine ρ_∞ we suppose $\tau=r+is$, and let s become infinite. Then q approaches zero and we get

$$\rho_\infty=(-1)^r.$$

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* Königsberger, loc. cit.

To determine ρ_0 we give to τ the value is, and we find* that

$$\rho_0 = (-1)^r.$$

In order to determine $\underset{a=1,\,2,\,\ldots\,,\,n-1}{\rho_a}$, we proceed as follows: Let

$$f(v,\,\varkappa) = 0$$

be the equation whose roots are

$$v_{\substack{\mu\\ \nu}} = \prod_{1,\,m}^{p} \operatorname{cn}(4p\varpi/k);$$

then

$$f\left\{(-1)^r \frac{\psi^n(\tau)}{\phi^n(\tau)} \cdot \frac{\phi\left(\dfrac{\tau}{n}\right)}{\psi\left(\dfrac{\tau}{n}\right)},\ \varkappa(\tau)\right\} = 0.$$

In this identity replace τ by $\tau + 16a$ and we get

$$f\left\{(-1)^r \frac{\psi^n(\tau)}{\phi^n(\tau)} \cdot \frac{\phi\left(\dfrac{\tau+16a}{n}\right)}{\psi\left(\dfrac{\tau+16a}{n}\right)},\ \varkappa(\tau)\right\} = 0.$$

which shows that

$$\rho_a = (-1)^r. \qquad a = 0,\,1,\,\ldots\,.\,,\,n-1)$$

Hence

$$\left.\begin{aligned} v_\infty &= (-1)^r \frac{\psi^n(\tau)}{\phi^n(\tau)} \cdot \frac{\phi(n\tau)}{\psi(n\tau)}, \\ \underset{a=0,\,1,\,\ldots\,.\,,\,n-1}{v_a} &= (-1)^r \frac{\psi^n(\tau)}{\phi^n(\tau)} \cdot \frac{\phi\left(\dfrac{\tau+16a}{n}\right)}{\psi\left(\dfrac{\tau+16a}{n}\right)}, \end{aligned}\right\} \tag{8}$$

or

$$\left.\begin{aligned} v_\infty &= (-1)^r.\,2^{-m}(1 + a_1 q + \ldots\,.), \\ \underset{a=0,\,1,\,\ldots\,.\,,\,n-1}{v_a} &= (-1)^r 2^{-m}.\,q^{-\frac{r}{n}} \varepsilon^a (1 + \varepsilon^{8a} b_1 q^{\frac{1}{n}} + \ldots\,.). \end{aligned}\right\} \tag{9}$$

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* Weber, "Ellipt. Funct.," p. 65.

III.—$w_{\frac{\mu}{\nu}}$.

We have*

$$\prod_{1,\,m}^{p} \frac{\vartheta_3(v-p\omega)\,\vartheta_3(v+p\omega)}{\vartheta_0(v-p\omega)\,\vartheta_0(v+p\omega)} = \frac{\vartheta_3}{\vartheta_0}\left(v_{\frac{\mu}{\nu}}/\tau_{\frac{\mu}{\nu}}\right).\frac{\vartheta_0}{\vartheta_3}(v/\tau). \tag{10}$$

Hence

$$\prod_{1,\,m}^{p} \frac{\vartheta_2^2}{\vartheta_0^2}(p\omega) = \frac{\vartheta_3}{\vartheta_0}\left(0/\tau_{\frac{\mu}{\nu}}\right).\frac{\vartheta_0}{\vartheta_3}(0/\tau)$$

and

$$\begin{aligned} w_{\frac{\mu}{\nu}}^2 &= (\sqrt{k'})^n.\frac{\vartheta_3}{\vartheta_0}\left(0/\tau_{\frac{\mu}{\nu}}\right) \\ &= \frac{\psi^{2n}(\tau)}{\psi^2\left(\tau_{\frac{\mu}{\nu}}\right)}. \end{aligned}$$

Hence

$$w_{\frac{\mu}{\nu}} = \rho_{\frac{\mu}{\nu}}\frac{\psi^n(\tau)}{\psi\left(\tau_{\frac{\mu}{\nu}}\right)}, \tag{11}$$

where $\rho_{\frac{\mu}{\nu}} = \pm 1$.

Reasoning as before, we get

$$\rho_\infty = 1, \quad \rho_a = (-1)^r. \qquad a = 0, 1, \ldots\ldots, n-1$$

Hence

$$\left.\begin{aligned} w_\infty &= \frac{\psi^n(\tau)}{\psi(n\tau)}, \\ \underset{a=0,1,\ldots\ldots,n-1}{w_a} &= (-1)^r\frac{\psi^n(\tau)}{\psi\left(\dfrac{\tau+16a}{n}\right)}, \end{aligned}\right\} \tag{12}$$

or

$$\left.\begin{aligned} w_\infty &= 1 + a_1 q + \ldots\ldots \\ \underset{a=0,1,\ldots\ldots,n-1}{w_a} &= (-1)^r\left\{1 + \varepsilon^{8a} b_1 q^{\frac{1}{n}} + \ldots\ldots\right\}. \end{aligned}\right. \tag{13}$$

Having thus obtained expressions for the three quantities $u_{\frac{\mu}{\nu}}$, $v_{\frac{\mu}{\nu}}$, $w_{\frac{\mu}{\nu}}$, we get the following expressions for $y_{\frac{\mu}{\nu}}$:

$$\left.\begin{aligned} y_\infty &= (-1)^{\beta r}.n^\alpha.\frac{\phi^{2\alpha+\beta}(n\tau)\;.\psi^{n(\beta+\gamma)}(\tau).\vartheta_3^{2\alpha}(0/n\tau)}{\phi^{n(2\alpha+\beta)}(\tau).\psi^{\beta+\gamma}(n\tau)\;.\vartheta_3^{2\alpha}(0/\tau)}, \\ \underset{a=0,1,\ldots\ldots,n-1}{y_a} &= (-1)^{m\alpha+r(\beta+\gamma)}.\frac{\phi^{2\alpha+\beta}\left(\dfrac{\tau+16a}{n}\right).\psi^{n(\beta+\gamma)}(\tau).\vartheta_3^{2\alpha}\left(0\middle/\dfrac{\tau+16a}{n}\right)}{\phi^{n(2\alpha+\beta)}(\tau).\psi^{\beta+\gamma}\left(\dfrac{\tau+16a}{n}\right).\vartheta_3^{2\alpha}(0/\tau)}, \end{aligned}\right\} \tag{14}$$

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* Königsberger, loc. cit.

or

$$\left.\begin{aligned} y_\infty &= (-1)^{\beta r} . n^\alpha . 2^{-m(2\alpha+\beta)}(1 + a_1 q + \dots .), \\ \underset{a=0,1,\dots,n-1}{y_a} &= (-1)^{m\alpha + r(\beta+\gamma)} . 2^{-m(2\alpha+\beta)} . \varepsilon^{a(2\alpha+\beta)} . q^{\frac{-r(2\alpha+\beta)}{n}} (1 + b_1 \varepsilon^{8a} q^{\frac{1}{n}} + \dots . \end{aligned}\right\} \quad (15)$$

2.—*A Superior Limit of the Degree of x.*

To determine this we have to determine the sum of the orders of the infinities of the roots for the infinite κ-plane. The only points in the κ-plane for which the roots may vanish or become infinite are $\kappa = 0, 1, \infty$. We will, therefore, determine the value of the roots for these three values of κ.

I. $\kappa = 0$. From (15) we get

$$\left.\begin{aligned} (y_\infty)_{\kappa=0} &= (-1)^{\beta r} . n^\alpha . 2^{-m(2\alpha+\beta)}, \\ (y_a)_{\kappa=0} &= \infty \text{ of order } \frac{r(2\alpha+\beta)}{n} . \end{aligned}\right\} \quad (16)$$

II. $\kappa = 1$. It is easy to show, on passing to exponentials, that

$$\left.\begin{aligned} (u_0)_{\kappa=1} &= (-1)^m . n, \\ (u_{\infty, 1, \dots, n-1})_{\kappa=1} &= 1, \end{aligned}\right\} \quad (17)$$

For $v_{\frac{\mu}{\nu}}$ we have

$$\begin{aligned} v_\infty(\tau) &= \prod_{1,m}^{p} \frac{1}{\operatorname{cn}\left(\frac{4piL'}{n} \middle| k'\right)} \\ &= \frac{1}{v_0(\tau_1)} \qquad \text{where } \tau_1 = -\frac{1}{\tau} \\ &= (-1)^r . 2^m q_1^{\frac{r}{n}} (1 + a_1 q_1^{\frac{1}{n}} + \dots .). \qquad q_1 = e^{\pi i \tau_{\frac{1}{\infty}}} \end{aligned} \quad (18)$$

Similarly we get

$$\begin{aligned} v_0(\tau) &= \frac{1}{v_\infty(\tau_1)} \\ &= (-1)^r \, 2^m (1 + b_1 q_1 + \dots .) \end{aligned} \quad (19)$$

and

$$\begin{aligned} \underset{a=1 \dots n-1}{v_a(\tau)} &= \frac{1}{v_b(\tau_1)} \\ &= (-1)^r . 2^m q_1^{\frac{r}{n}} \varepsilon^{-b} (1 + c_1 \varepsilon^{8b} q_1^{\frac{1}{n}} + \dots .). \end{aligned} \quad (20)$$

Hence

$$\left.\begin{aligned}(v_0)_{\kappa=1} &= (-1)^r . 2^m,\\ (v_{\infty,1\,\ldots\,n})_{\kappa=1} &= 0 \text{ of order } \frac{r}{n}.\end{aligned}\right\} \tag{21}$$

For $w_{\frac{\mu}{\nu}}$ we obtain by a similar reasoning

$$\left.\begin{aligned}w_\infty(\tau) &= \frac{w_0(\tau_1)}{v_0(\tau_1)} = 2^m . q_1^{\frac{r}{n}}(1 + a_1 q_1^{\frac{1}{n}} + \ldots\ldots),\\ w_0(\tau) &= \frac{w_\infty(\tau_1)}{v_\infty(\tau_1)} = (-1)^r . 2^m (1 + b_1 q_1 + \ldots\ldots),\\ \underset{a=1\,\ldots\,n-1}{w_a(\tau_1)} &= \frac{w_b(\tau_1)}{v_b(\tau_1)} = 2^m . q_1^{\frac{r}{n}} . \varepsilon^{-b}(1 + c_1 \varepsilon^{8b} q^{\frac{1}{n}} + \ldots\ldots),\end{aligned}\right\} \tag{22}$$

or

$$\left.\begin{aligned}(w_0)_{\kappa=1} &= (-1)^r . 2^m,\\ (w_{\infty,1\,\ldots\,n-1})_{\kappa=1} &= 0 \text{ of order } \frac{r}{n}.\end{aligned}\right\} \tag{23}$$

Hence from (17), (21) and (23),

$$\left.\begin{aligned}(y_0)_{\kappa=1} &= (-1)^{ma + r(\beta+\gamma)} . n^a . 2^{m(\beta+\gamma)},\\ (y_{\infty,1,\,\ldots\,,n-1})_{\kappa=1} &= 0 \text{ of order } \frac{r}{n}(\beta+\gamma),\end{aligned}\right\} \tag{24}$$

III. $\varkappa = \infty$

We have

$$u_\infty(\tau) = \lambda^{2m} \prod_{1,\,m}^{p} \operatorname{sn}^2\left(4p\,\frac{L + iL'}{n}\Big/\lambda\right) = u_a(\tau_1),$$

where $\lambda = \dfrac{1}{k}$, $\tau_1 = \dfrac{\tau}{1-\tau}$, $16a \equiv 1 \bmod n$.

Similarly,

$$u_a(\tau) = \lambda^{2m} u_\infty(\tau_1) \text{ if } 16a \equiv -1 \bmod n,$$
$$u_a(\tau) = \lambda^{2m} u_b(\tau_1) \text{ if } (16b - 1)a \equiv -b \bmod n.$$

Hence

$$\left.\begin{aligned}u_\infty(\tau) &= (-1)^m . 2^{2m} q_1^{m - \frac{2r}{n}} . \varepsilon^{2a}(1 + a_1 \varepsilon^{8a} q_1^{\frac{1}{n}} + \ldots\ldots),\\ &\qquad \text{where } 16a \equiv 1 \bmod n,\\ u_a(\tau) &= n . 2^{2m} q_1^m (1 + b_1 q_1 + \ldots\ldots),\\ &\qquad \text{where } 16a \equiv -1 \bmod n,\\ u_a(\tau) &= (-1)^m . 2^{2m} q_1^{m - \frac{2r}{n}} . \varepsilon^{2b}(1 + c_1 \varepsilon^{8b} q_1^{\frac{1}{n}} + \ldots\ldots),\\ &\qquad \text{where } (16b - 1)a \equiv -b \bmod n.\end{aligned}\right\} \tag{25}$$

In the same way we get

$$\left.\begin{aligned}
v_\infty(\tau) = w_a(\tau) &= (-1)^r\left(1 + a_1\varepsilon^{8a}q_1^{\frac{1}{n}} + \ldots .\right), \\
&\quad \text{where } 16a \equiv 1 \bmod n, \\
v_a(\tau) = w_\infty(\tau_1) &= 1 + b_1 q_1 + \ldots . . \\
&\quad \text{where } 16a \equiv -1 \bmod n, \\
v_a(\tau) = w_b(\tau_1) &= (-1)^r\left(1 + c_1\varepsilon^{8b}q_1^{\frac{1}{n}} + \ldots , \right. \\
&\quad \text{where } (16b - 1)\,a \equiv -b \bmod n.
\end{aligned}\right\} \tag{26}$$

and

$$\left.\begin{aligned}
w_\infty(\tau) = v_a(\tau_1) &= (-1)^r . 2^{-m} q_1^{-\frac{r}{n}} . \varepsilon^a\left(1 + a_1\varepsilon^{8a}q_1^{\frac{1}{n}} + \ldots .\right), \\
&\quad \text{where } 16a \equiv 1 \bmod n, \\
w_a(\tau) = v_\infty(\tau_1) &= (-1)^r . 2^{-m}\left(1 + b_1 q_1 + \ldots .\right), \\
&\quad \text{where } 16a \equiv -1 \bmod . \, n, \\
w_a(\tau) = v_b(\tau_1) &= (-1)^r . 2^{-m} q_1^{-\frac{r}{n}} . \varepsilon^b\left(1 + c_1\varepsilon^{8b}q_1^{\frac{1}{n}} + \ldots .\right), \\
&\quad \text{where } (16b - 1)\,a \equiv -b \bmod n.
\end{aligned}\right\} \tag{27}$$

Hence

$$\left.\begin{aligned}
(y_a)_{\kappa=\infty} &= 0 \text{ of order } \alpha m, \text{ if } 16a \equiv -1 \bmod n, \\
(y_{\infty,\,a})_{\kappa=\infty} &= 0 \text{ of order } \alpha m - \frac{r}{n}(2\alpha + \gamma), \text{ if } 16a \not\equiv -1 \bmod n.
\end{aligned}\right\} \tag{28}$$

3.—*Equations that can be derived from a given Equation of Transformation by Means of Linear Transformations.*

By applying a linear transformation to the equation whose roots are $y_{\frac{\mu}{\nu}}$ we obtain other equations of transformation. The results we obtain are given in the table below, where to the left the substitution to be made and to the right the roots of the resulting equation are given. $\varkappa'$ is defined by $\varkappa + \varkappa' = 1$.

1. $\begin{pmatrix} \varkappa & , & y \\ -\dfrac{\varkappa}{\varkappa'} & , & \varkappa'^{\alpha m} y \end{pmatrix}, \quad \prod_{1,\,m}^{p} \operatorname{sn}^{2\alpha} . \operatorname{cn}^{\beta} . \operatorname{dn}^{-(2\alpha+\beta+\gamma)}(4p\varpi/k).$

2. $\begin{pmatrix} \varkappa, & y \\ \varkappa', & (-1)^{\alpha m} y \end{pmatrix}, \quad \prod_{1,\,m}^{p} \operatorname{sn}^{2\alpha} . \operatorname{cn}^{-(2\alpha+\beta+\gamma)} . \operatorname{dn}^{\gamma}(4p\varpi/k).$

$$3.\quad \begin{pmatrix} \varkappa \,, & y \\ -\dfrac{\varkappa'}{\varkappa}\,, & (-1)^{\alpha m}.\varkappa^{\alpha m}.y \end{pmatrix},\quad \prod_{1,\,m}^{p} \mathrm{sn}^{2\alpha}.\,\mathrm{cn}^{\gamma}.\,\mathrm{dn}^{-(2\alpha+\beta+\gamma)}\,(4p\varpi/k).$$

$$4.\quad \begin{pmatrix} \varkappa \,, & y \\ \dfrac{1}{\varkappa'}\,, & (-1)^{\alpha m}.\varkappa'^{\alpha m}.y \end{pmatrix},\quad \prod_{1,\,m}^{p} \mathrm{sn}^{2\alpha}.\,\mathrm{cn}^{-(2\alpha+\beta+\gamma)}.\,\mathrm{dn}^{\beta}\,(4p\varpi/k).$$

$$5.\quad \begin{pmatrix} \varkappa \,, & y \\ \dfrac{1}{\varkappa}\,, & \varkappa^{\alpha m}.y \end{pmatrix},\quad \prod_{1,\,m}^{p} \mathrm{sn}^{2\alpha}.\,\mathrm{cn}^{\gamma}.\,\mathrm{dn}^{\beta}\,(4p\varpi/k).$$

Hence, using the notation

$$y_{\frac{\mu}{\nu},\,\alpha,\,\beta,\,\gamma} = \prod_{1,\,m}^{p} \mathrm{sn}^{2\alpha}.\,\mathrm{cn}^{\beta}.\,\mathrm{dn}^{\gamma}\,(4p\varpi/k),$$

the substitutions given above lead to equations whose roots are

1. $y_{\frac{\mu}{\nu},\,\alpha,\,\beta,\,-(2\alpha+\beta+\gamma)}$,
2. $y_{\frac{\mu}{\nu},\,\alpha,\,-(2\alpha+\beta+\gamma),\,\gamma}$,
3. $y_{\frac{\mu}{\nu},\,\alpha,\,\gamma,\,-(2\alpha+\beta+\gamma)}$,
4. $y_{\frac{\mu}{\nu},\,\alpha,\,-(2\alpha+\beta+\gamma),\,\beta}$,
5. $y_{\frac{\mu}{\nu},\,\alpha,\,\gamma,\,\beta}$

respectively.

4.—*Applications.*

I. *The equation whose roots are*

$$y_{\frac{\mu}{\nu}} = \prod_{1,\,m}^{p} \frac{\mathrm{dn}^2}{\mathrm{cn}}\,(4p\varpi/k).$$

From (15) we get

$$\left.\begin{aligned} y_{\infty} &= (-1)^{r}.\,2^{m}\,(1+\ldots.), \\ \underset{a=0,\,1,\,\ldots.,\,n-1}{y_{a}} &= (-1)^{r}.\,2^{m}.\,\varepsilon^{-a'}.\,q^{\frac{r}{n}}(1+\ldots.), \end{aligned}\right\} \qquad (29)$$

and from (29) we get the values of the roots for $\kappa = 0,\ 1,\ \infty$ as follows:

$$\left.\begin{array}{ll}
\kappa = 0 & \left\{\begin{array}{ll} y_\infty & = (-1)^r \,.\, 2^m, \\ \underset{a=0,\,1,\,\ldots,\,n-1}{y_a} & = 0 \text{ of order } \dfrac{r}{n}, \end{array}\right. \\
\kappa = 1 & \left\{\begin{array}{ll} y_0 & = (-1)^r \,.\, 2^m, \\ \underset{a=1,\,\ldots,\,n-1}{y_{\infty,\,a}} & = 0 \text{ of order } \dfrac{r}{n}, \end{array}\right. \\
\kappa = \infty & \left\{\begin{array}{ll} y_a & = 2^{-2m}, \text{ if } 16a \equiv -1 \bmod n, \\ y_{\infty,\,a} & = \infty \text{ of order } \dfrac{2r}{n} \text{ if } 16a \not\equiv -1 \bmod n. \end{array}\right.
\end{array}\right\} \tag{30}$$

Since the equation remains unchanged if we replace κ by κ', it must be of the form

$$y^{n+1} + f_1(\kappa\kappa')\, y^n + \ldots\ldots + f_{n+1}(\kappa\kappa') = 0, \tag{31}$$

where

$$f_s(\kappa\kappa') = b_{s,\,0} + b_{s,\,1}\kappa\kappa' + \ldots\ldots + b_{s,\,m_s}(\kappa\kappa')^{m_s} \text{ and } m_s \leqq r.$$

The term $f_{n+1}(\kappa\kappa')$ being equal to the product of the roots, can easily be determined, and we get

$$f_{n+1}(\kappa\kappa') = (\kappa\kappa')^r. \tag{32}$$

For $\kappa = 0$, n roots vanish. Hence $f_2,\ f_3,\ \ldots\ldots,\ f_n$ must all contain the factor $\kappa\kappa'$ and also

$$b_{1,\,0} = (-1)^{r+1} \,.\, 2^m. \tag{33}$$

For $\kappa = \infty$, n roots become infinite. Hence we see that

$$\begin{array}{l} m_s < r, \quad s = 1,\ 2,\ \ldots\ldots,\ n-1, \\ m_n = r, \end{array} \quad \text{and} \quad b_{n,\,m_n} = -2^{2m}. \tag{34}$$

For $n = 3$ the equation is

$$y^4 + 2y^3 - 4\kappa\kappa' y + \kappa\kappa' = 0. \tag{35}$$

From this equation we derive two other equations by the substitutions

$$\begin{pmatrix} \kappa\ , & y \\ -\dfrac{\kappa}{\kappa'}\ , & y \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \kappa\ , & y \\ -\dfrac{\kappa'}{\kappa}\ , & y \end{pmatrix},$$

as shown in the table below:

ROOTS.	EQUATIONS.
$\prod_{1,\,m}^{p} \frac{\text{dn}^2}{\text{cn}}(4p\varpi/k)$	$y^4 + 2y^3 - 4\varkappa\varkappa' y + \varkappa\varkappa' = 0$
$\prod_{1,\,m}^{p} \frac{1}{\text{cn}\,.\,\text{dn}}(4p\varpi/k)$	$y^4 + 2y^3 + \frac{4\varkappa}{\varkappa'^2} y - \frac{\varkappa}{\varkappa'^2} = 0$
$\prod_{1,\,m}^{p} \frac{\text{cn}^2}{\text{dn}}(4p\varpi/k)$	$y^4 + 2y^3 + \frac{4\varkappa'}{\varkappa^2} y - \frac{\varkappa'}{\varkappa^2} = 0$

II. *The equation whose roots are*

$$y_{\frac{\mu}{\nu}} = \prod_{1,\,m}^{p} \frac{\text{cn}}{\text{sn}^2}(4p\varpi/k).$$

From (15) we get

$$\left.\begin{aligned} y_\infty &= \frac{(-1)^r . 2^m}{n}(1 + \dots .), \\ \underset{a-0,\,1,\,\dots,\,n-1}{y_a} &= (-1)^{r-m} . 2^m . \varepsilon^{-a} . q^{\frac{r}{n}}(1 + \dots .), \end{aligned}\right\} \tag{36}$$

and from (29),

$$\left.\begin{aligned} &\varkappa = 0 \left\{\begin{aligned} y_\infty &= \frac{(-1)^r . 2^m}{n}, \\ \underset{a=0,\,1,\,\dots,\,n-1}{y_a} &= 0 \text{ of order } \frac{r}{n}, \end{aligned}\right. \\ &\varkappa = 1 \left\{\begin{aligned} y_0 &= \frac{(-1)^{r-m} . 2^m}{n}, \\ \underset{a=1,\,\dots,\,n-1}{y_{\infty,\,a}} &= 0 \text{ of order } \frac{r}{n}, \end{aligned}\right. \\ &\varkappa = \infty \left\{\begin{aligned} y_a &= \infty \text{ of order } m, && 16a \equiv -1 \bmod n \\ y_{\infty,\,a} &= \infty \text{ of order } -\frac{2r}{n} + m, && 16a \not\equiv -1 \bmod n \end{aligned}\right. \end{aligned}\right\} \tag{37}$$

If we replace $\varkappa$ by $\varkappa'$, the equation is transformed into an equation whose roots are

$$(-1)^m \prod_{1,\,m}^{p} \frac{\text{cn}}{\text{sn}^2}(4p\varpi/k).$$

Hence if m is even, the equation must be of the form

$$y^{n+1} + f_1(\varkappa\varkappa')\,y^n + \ldots\ldots + f_{n+1}(\varkappa\varkappa') = 0, \tag{38}$$

and if m is odd,

$$y^{n+1} + (\varkappa - \varkappa')f_1(\varkappa\varkappa')\,y^n + f_2(\varkappa\varkappa')\,y^{n-1} + \ldots\ldots + f_{n+1}(\varkappa\varkappa') = 0, \tag{39}$$

where

$$f_s(\varkappa\varkappa') = b_{s,\,0} + b_{s,\,1}\varkappa\varkappa' + \ldots\ldots + b_{s,\,m_s}(\varkappa\varkappa')^{m_s}.$$

When m is even, we have $m_s \leqq r$, and when m is odd, we have

$$m_s \leqq \begin{cases} r & \text{if } s \text{ is even,} \\ r-1 & \text{if } s \text{ is odd.} \end{cases}$$

For $f_{n+1}(\varkappa\varkappa')$ we get the expression

$$f_{n+1}(\varkappa\varkappa') = \frac{(-1)^m}{n}(\varkappa\varkappa')^r. \tag{40}$$

The coefficients $f_2, \ldots\ldots, f_n$ must all contain the factor $\varkappa\varkappa'$, and we also get

$$b_{1,\,0} = \frac{(-1)^{r+1}.\,2^m}{n}. \tag{41}$$

From the above, and making use of the fact that for $\varkappa = \infty$ all the roots become infinite, we obtain the equation for $n = 3$. This equation and those that can be derived from it by the substitutions

$$\begin{pmatrix} \varkappa\,, & y \\ -\dfrac{\varkappa}{\varkappa'}\,, & \varkappa'^{-m}y \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \varkappa\,, & y \\ -\dfrac{\varkappa'}{\varkappa}\,, & (-1)^m\varkappa^{-m}y \end{pmatrix}$$

are given below.

ROOTS.	EQUATIONS.
$\prod\limits_{1,\,m}^{p}\dfrac{\text{cn}}{\text{sn}^2}(4p\varpi/k)$	$y^4 - \frac{2}{3}(\varkappa - \varkappa')\,y^3 - \dfrac{\varkappa\varkappa'}{3} = 0$
$\prod\limits_{1,\,m}^{p}\dfrac{\text{cn}\,.\,\text{dn}}{\text{sn}^2}(4p\varpi/k)$	$y^4 + \frac{2}{3}(1 + \varkappa)\,y^3 + \dfrac{\varkappa\varkappa'^2}{3} = 0$
$\prod\limits_{1,\,m}^{p}\dfrac{\text{dn}}{\text{sn}^2}(4p\varpi/k)$	$y^4 - \frac{2}{3}(1 + \varkappa')\,y^3 + \dfrac{\varkappa'\varkappa^2}{3} = 0$

III. *The equation whose roots are*

$$y_{\frac{\mu}{\nu}} = \prod_{1,\, m}^{p} \mathrm{sn}^2 . \mathrm{cn} . \mathrm{dn}\,(4p\varpi/k).$$

From (15) we get

$$\left.\begin{aligned} y_\infty &= (-1)^r n . 2^{-3m}(1 + \ldots.), \\ y_a &= (-1)^m . 2^{-3m} . \varepsilon^{3a} . q^{-\frac{3r}{n}}(1 + \ldots.), \end{aligned}\right\} \tag{42}$$

and from (29),

$$\left.\begin{aligned} &\varkappa = 0 \begin{cases} y_\infty & = (-1)^r . n . 2^{-3m}, \\ \underset{a = 0,\, 1,\, \ldots.,\, n-1}{y_a} & = \infty \text{ of order } \dfrac{3r}{n}, \end{cases} \\ &\varkappa = 1 \begin{cases} y_0 & = (-1)^m . n . 2^{2m}, \\ \underset{a = 1,\, \ldots.,\, n-1}{y_{\infty,\, a}} & = 0 \text{ of order } \dfrac{2r}{n}, \end{cases} \\ &\varkappa = \infty \begin{cases} y_a & = 0 \text{ of order } m, & 16a \equiv -1 \bmod n \\ y_{\infty,\, a} & = 0 \text{ of order } \dfrac{mn - 3r}{n}. & 16a \not\equiv -1 \bmod n \end{cases} \end{aligned}\right\} \tag{43}$$

The equation must be of the form

$$y^{n+1} + \frac{g_1(\varkappa)}{\varkappa^{r_1}} y^n + \ldots. + \frac{g_n(\varkappa)}{\varkappa^{r_n}}\, y + \frac{A . \varkappa'^{\rho}}{\varkappa^{r_{n+1}}} = 0,$$

where $A =$ constant and $g_s(\varkappa) = a_{s,\,0} + a_{s,\,1}\varkappa + \ldots. + a_{s,\,m_s}\varkappa^{m_s}$, and it is also seen that the degree of $\varkappa$ cannot exceed $3r$.

The last term being the product of the roots, we get

$$r_{n+1} = 3r, \quad A = (-1)^{r+m} . n. \tag{44}$$

We have also

$$\frac{g_n(\varkappa)}{\varkappa^{r_n}} = -y_\infty \cdot y_0 \ldots. y_{n-1}\left(\frac{1}{y_\infty} + \ldots. + \frac{1}{y_{n-1}}\right),$$

which gives

$$r_n = 3r, \quad a_{n,\,0} = (-1)^{m+1} . 2^{3m}. \tag{45}$$

In order to determine r_1 and a superior limit of $r_2, r_3, \ldots., r_{n-1}$, we use the method employed by Pierpont.* Setting

$$S_p = y_\infty^p + y_0^p + \ldots. + y_{n-1}^p,$$

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* "On Modular Equations."

and denoting the order of infinity of S_p by σ_p, we get

$$\sigma_p = \frac{3pr - \nu_p}{n}, \qquad p = 1.\ 2, \ldots., n-1 \tag{46}$$

where ν_p is the smallest positive root of

$$8\nu + 3p \equiv 0 \mod n,$$

and hence

$$r_1 = \sigma_1,$$

and if $\sigma_1, \sigma_2, \ldots.$ form a regular sequence,* we also get

$$r_p \leqq \sigma_p. \qquad p = 2, 3, \ldots., n-1 \tag{47}$$

In order to determine ρ, m_1, m_n and a superior limit of $m_2, m_3, \ldots., m_{n-1}$, we make use of the fact that the equation remains unchanged for the substitution

$$\begin{pmatrix} x, & y \\ \frac{1}{x}, & x^m y \end{pmatrix},$$

From this fact we get the relations

$$\rho = 2r, \quad m_p = 2r_p - pm. \qquad (p = 1, 2, \ldots., n),\ a_{a,\lambda} = a_{a, m_a - \lambda} \tag{48}$$

For $x = 1$, n roots vanish and one root $= (-1)^m n \,.\, 2^{2m}$. Hence

$$\left.\begin{array}{l} a_{1,0} + a_{1,1} + \ldots. + a'_{1,m_1} = (-1)^{m+1}.\, n \,.\, 2^{2m}, \\ a_{2,0} + a_{2,1} + \ldots. + a_{2,m_2} = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n,0} + a_{n,1} + \ldots. + a_{n,m_n} = 0. \end{array}\right\} \tag{49}$$

The relations given above show that for $n = 3$ the equation is

$$y^4 + \frac{6(1+x)}{x} y^3 + \frac{a_{2,0}(1 - 2x + x^2)}{x^2} y^2 + \frac{8(1 - x - x^2 + x^3)}{x^3} y + \frac{3x'^2}{x^3} = 0.$$

But for $x = \infty$ the equation reduces to

$$y(y+2)^3 = 0.$$

Hence $$a_{2,0} = 12.$$

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* Pierpont, l. c.

From this equation we derive two other equations whose roots are

$$\prod_{1,\,m}^{p} \frac{\operatorname{sn}^2 \operatorname{dn}}{\operatorname{cn}^4} (4p\varpi/k) \quad \text{and} \quad \prod_{1,\,m}^{p} \frac{\operatorname{sn}^2 \operatorname{cn}}{\operatorname{dn}^4} (4p\varpi/k)$$

by the substitutions

$$\begin{pmatrix} \varkappa & y \\ \varkappa' & (-1)^m y \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \varkappa\,, & y \\ -\dfrac{\varkappa}{\varkappa'}\,, & \varkappa'^m y \end{pmatrix} \quad \text{resp.}$$

Thus we have for $n = 3$:

ROOTS.	EQUATIONS.
$\prod_{1,\,m}^{p} \operatorname{sn}^2 . \operatorname{cn} . \operatorname{dn} (4p\varpi/k)$	$y^4 + \frac{6(1+\varkappa)}{\varkappa} y^3 + \frac{12\varkappa'^2}{\varkappa^2} y^2 + \frac{8\varkappa'^2(1+\varkappa)}{\varkappa^3} y + \frac{3\varkappa'^2}{\varkappa^3} = 0$
$\prod_{1,\,m}^{p} \frac{\operatorname{sn}^2 \operatorname{dn}}{\operatorname{cn}^4} (4p\varpi/k)$	$y^4 - \frac{6(1+\varkappa')}{\varkappa'} y^3 + \frac{12\varkappa^2}{\varkappa'^2} y^2 - \frac{8\varkappa^2(1+\varkappa')}{\varkappa'^3} y + \frac{3\varkappa^2}{\varkappa'^3} = 0$
$\prod_{1,\,m}^{p} \frac{\operatorname{sn}^2 \operatorname{cn}}{\operatorname{dn}^4} (4p\varpi/k)$	$y^4 + \frac{6(\varkappa-\varkappa')}{\varkappa\varkappa'} y^3 + \frac{12}{\varkappa^2\varkappa'^2} y^2 + \frac{8(\varkappa-\varkappa')}{\varkappa^3\varkappa'^3} - \frac{3}{\varkappa^3\varkappa'^3} = 0$

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